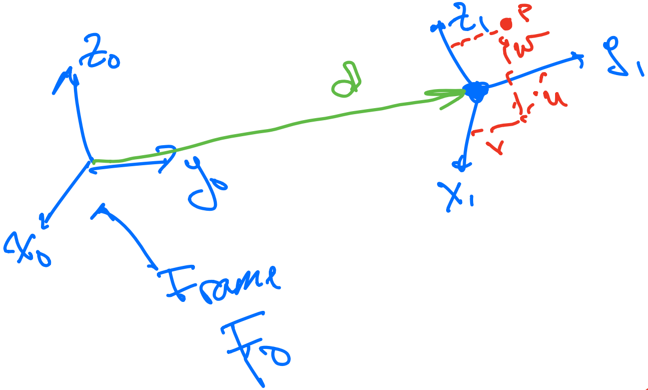


$$R_1^0 = \left[\, x_1 \cdot F_0 \;\middle|\; y_1 \cdot F_0 \;\middle|\; z_1 \cdot F_0 \,\right]$$

$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$$\left[\, u x_1 \cdot F_0 \;\middle|\; v y_1 \cdot F_0 \;\middle|\; w z_1 \cdot F_0 \,\right] = \left[\, x_1^0 \quad y_1^0 \quad z_1^0 \,\right]$$

$$\Downarrow$$

$$P^0 = R_1^0 \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_1^0 P^1$$

For the case of <u>No Translation</u>

Rotation Matrices

$$R \in SO(n) \quad , \quad c_i = i^{th} \text{ column of } R$$

$3 - c_i^T c_i = 1 \qquad c_i \cdot c_i = 1$

$3 - c_i^T c_j = 0 \qquad i \neq j \qquad c_i \cdot c_j = 0$

orthogonal Matrices

$\Rightarrow R^{-1} = R^T$

$\Downarrow$

$\cancel{3} \rightarrow \det R = +1$ $\rightarrow$ special $\Rightarrow$ special orthogonal

$SO(n)$ for $n \times n$ case

$R \in \mathbb{R}^{3 \times 3}$ for $n = 3$

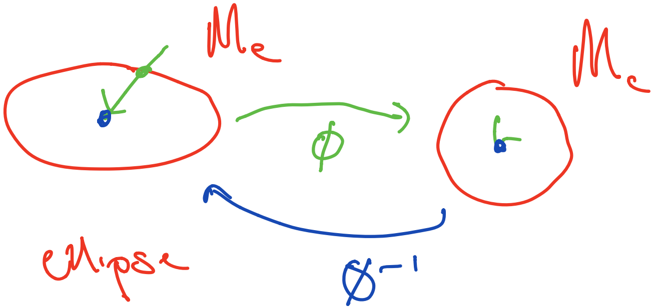
$\hookrightarrow \mathbb{R}^9$

$* \; n \in \{2, 3\}$ for us

Def if $\phi: U \to V$ a bijection & ϕ and ϕ^{-1} are cont., C^∞ (smooth)

ϕ is a homeomorphism/diffeomorphism

Thus U & V are homeomorphic/diffeomorphic.



$$\phi = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

↳ smooth on M_e + M_c

ϕ is global diffeomorphism

Def: a neighborhood U of pt x_0 can be defined

as $U = \{ x \mid d(x, x_0) < \epsilon \}$, $\epsilon > 0$

distance

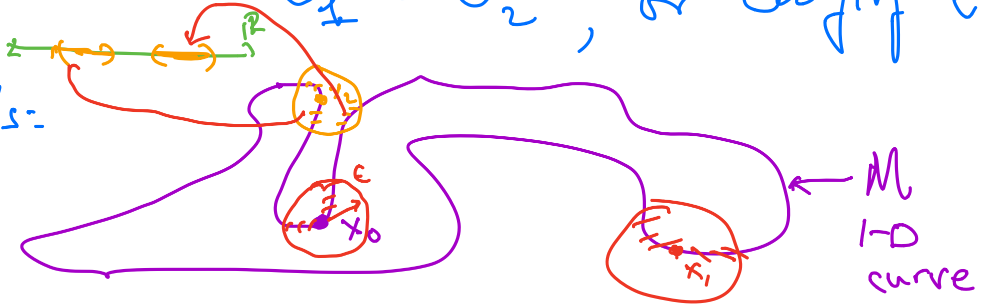
strict inequality

$\Longrightarrow$ Build other neighborhoods by $U_1 \cup U_2$, or changing ϵ

Second method:

Induced neighborhoods:

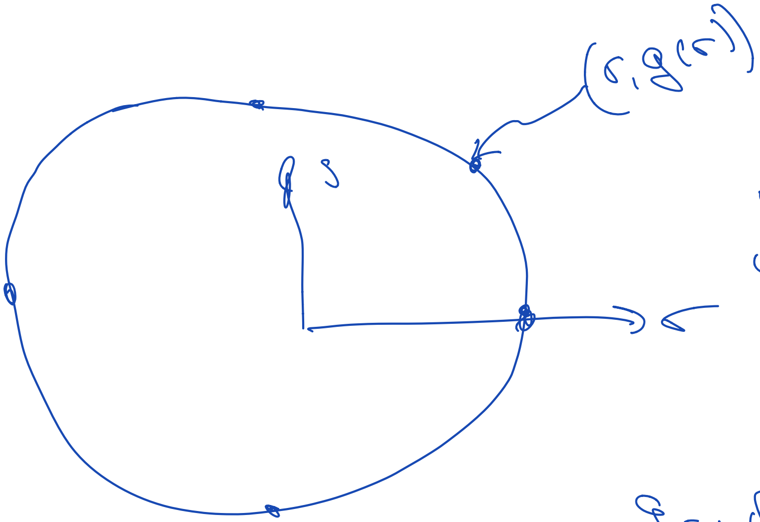
intersect U above with M.



Def: A set S is a k-dimensional smooth manifold if it is locally homeomorphic diffeomorphic to $\mathbb{R}^k$, i.e., $q \in S$ there exists a neighborhood $U \subset S$ with $q \in U$ s.t. U is homeomorphic diffeomorphic to $\mathbb{R}^k$.

M_e

No global ϕ from M_e to $\mathbb{R}^1$



$$\underbrace{r^2+s^2-1=0}_{f(r,s)=0}$$

$$s=+\sqrt{1-r^2}$$

Implicit function thm

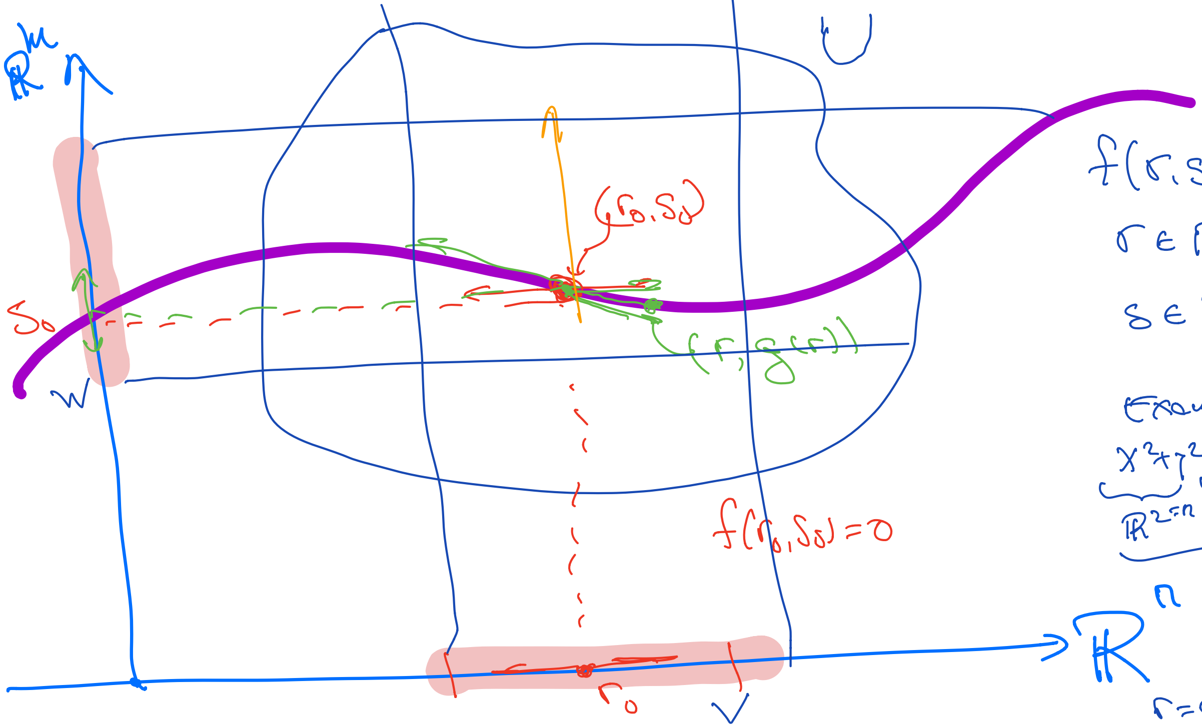
$U \subset \mathbb{R}^n \times \mathbb{R}^m$ an open set, $f: U \to \mathbb{R}^m$ is a C^∞ fn.

Let $r_0 \in \mathbb{R}^n$, $s_0 \in \mathbb{R}^m$ and $\boxed{f(r_0, s_0) = 0}$.

If $\det J(r_0, s_0) \neq 0$, then there exists open neighborhood V of r_0, $V \subset \mathbb{R}^n$ and an open neighborhood W of s_0, $W \subset \mathbb{R}^m$ s.t. $V \times W \subset U$ and $\exists g: V \to W$ s.t. g is C^∞

and $\boxed{f(r, g(r)) = 0}$ for $(r, s = g(r)) \in V \times W$

$$J = \left[\frac{\partial f}{\partial s} \right]$$



$f(r, s) = 0$

$r \in \mathbb{R}^n$

$s \in \mathbb{R}^m$

Example:

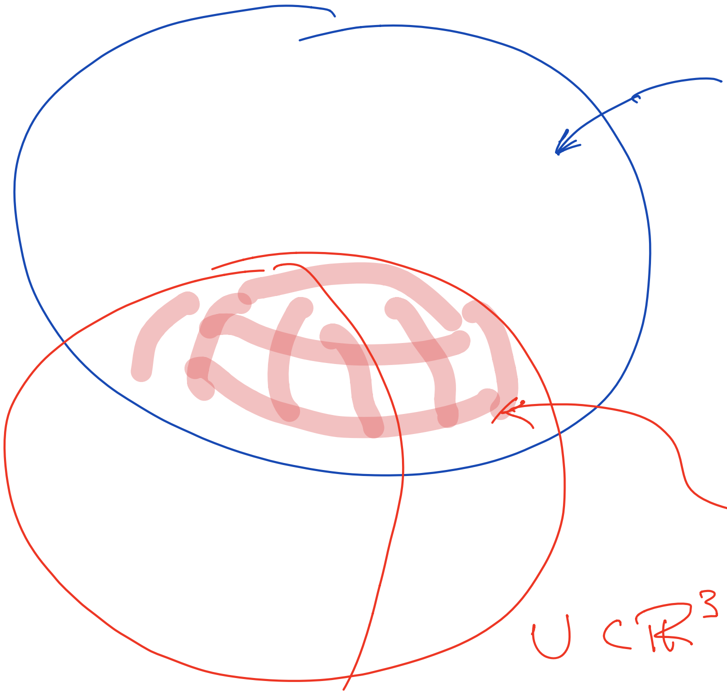
$$\underbrace{\underbrace{x^2 + y^2}_{\mathbb{R}^2 = n} + \underbrace{z^2}_{\mathbb{R}^1 = m} - 4}_{} = 0$$

$f \downarrow$

$r = (x, y)$

$s = z$

Space has dimension $n + m$



sphere

$$\underbrace{x^2+y^2}_{x,\,y} + \underbrace{z^2}_{z} - 4 = 0$$

$$\sigma \qquad s$$

$$f(r,s) = 0$$

intersection of sphere with a neighborhood $U \subset \mathbb{R}^3$

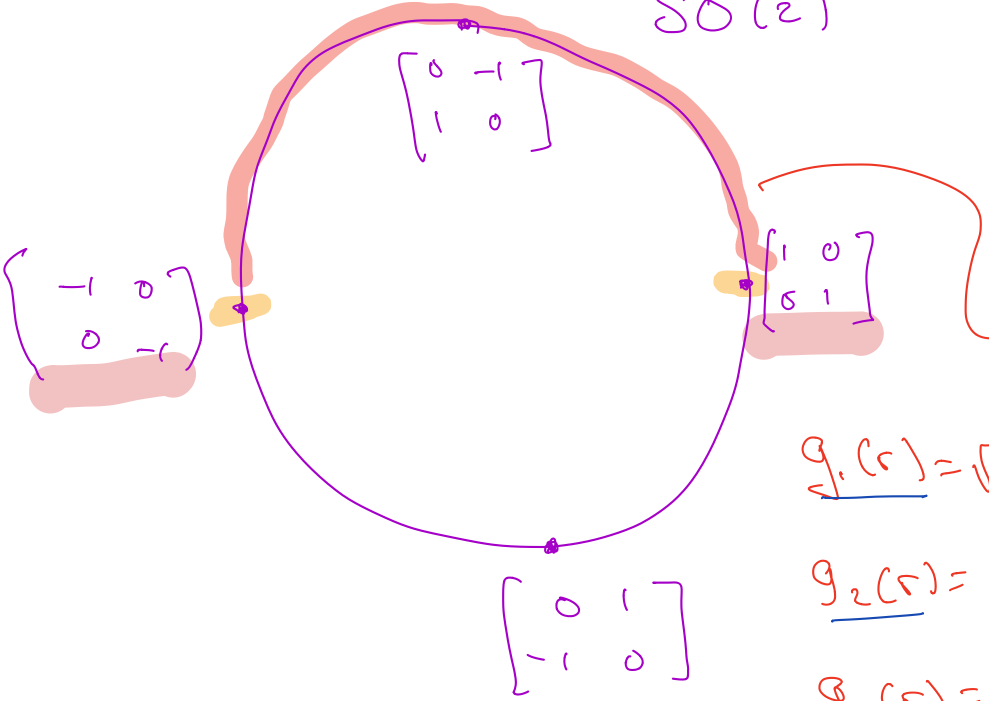
$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \rightsquigarrow \begin{bmatrix} r & s_1 \\ s_2 & s_3 \end{bmatrix} \Rightarrow \begin{bmatrix} s_1 & r \\ s_2 & s_3 \end{bmatrix}$$

$$\left.\begin{aligned} f_1 &= rs_1 + s_2 s_3 = 0 \\ f_2 &= r^2 + s_2^2 - 1 = 0 \\ f_3 &= s_1^2 + s_3^2 - 1 = 0 \end{aligned}\right\} \frac{\partial f}{\partial s} = \begin{array}{ccc} \underline{r} & \underline{s_3} & \underline{s_4} \\ \left[\begin{matrix} 0 \\ 2s_1 \end{matrix}\right. & \begin{matrix} 2s_2 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 2s_3 \end{matrix}\right] \end{array}$$

$$\det J = 4s_2 [\underbrace{rs_3 - s_1 s_2}_{\det R = \pm 1}] = \boxed{4s_2}$$

$$\Rightarrow \boxed{\text{Need } s_2 \neq 0}$$

SO(2)



Choose

$$R = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

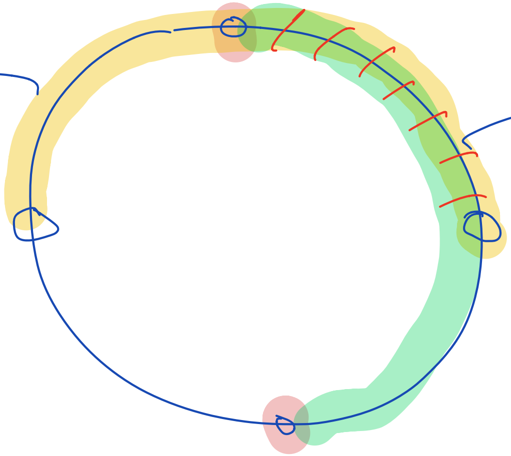
$$g(r) = (s_1, s_2, s_3)$$

$$g_1(r) = \sqrt{1-r^2} = s_1$$

$$g_2(r) = -\sqrt{1-r^2} = s_2$$

$$g_3(r) = r = s_3$$

$$R = \begin{bmatrix} s_1 & s_2 \\ r & s_3 \end{bmatrix} \longrightarrow \det J \neq 0 \Rightarrow s_1 \neq 0$$



$g^b_1(r) = s_1$

$g^b_1(r) = s_2$

$g^b(r) = s_3$

$g^a_1(r) = s_1$

$g^a_2(r) = s_2$

$g^a_3(r) = s_3$

Def A chart is a pair (U, ϕ) where $U \subseteq M$ is an open set and ϕ is a diffeomorphism $\phi : U \longrightarrow X \subseteq \mathbb{R}^n$.

$U = \{ R \in SO(2) \mid -1 < r < 1,\ 0 < s_1 < -1,\ -1 < s_2 < 0,\ -1 < s_3 < 1 \}$

EXAMPLE

$R = \begin{bmatrix} r & s_1 \\ s_2 & s_3 \end{bmatrix}$

$\phi^{-1} : \mathbb{R} \to SO(2)$ is called a parameterization

$R = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$

$R \in SO(2)$

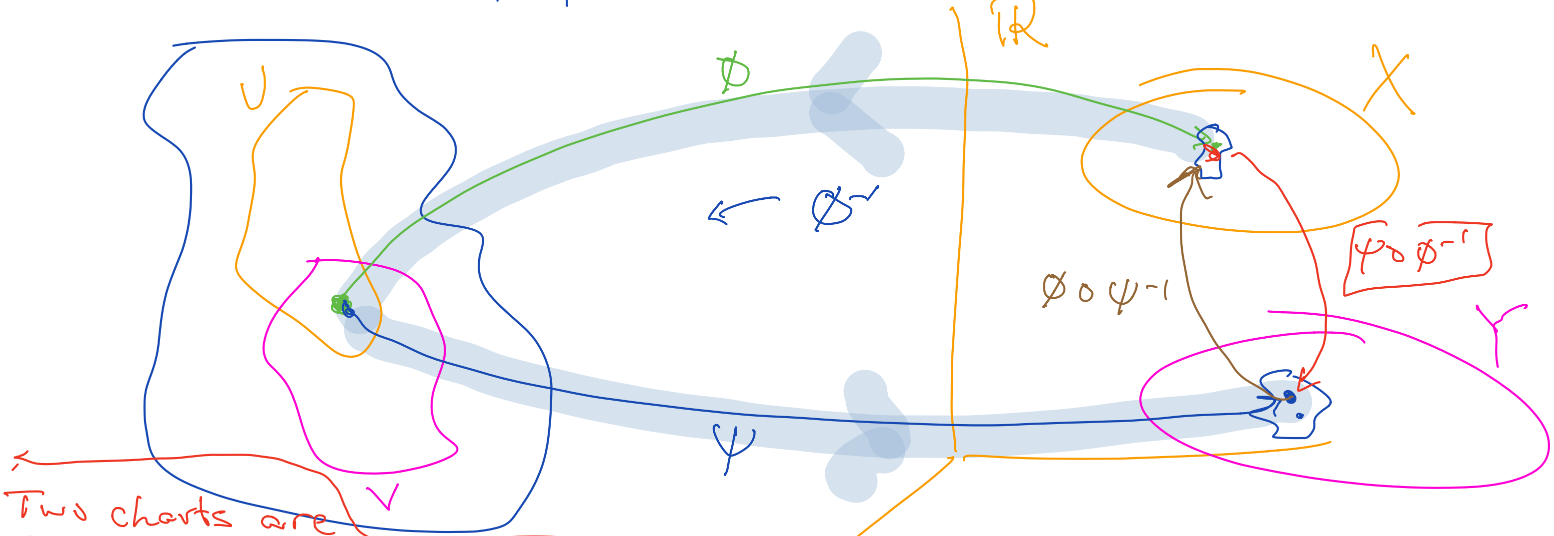
$\phi : SO(2) \to \mathbb{R}^1$

$\phi(R) = \cos^{-1}(r)$

$\hookrightarrow$ coordinate map

This example is NOT an example of the implicit fn Thm !

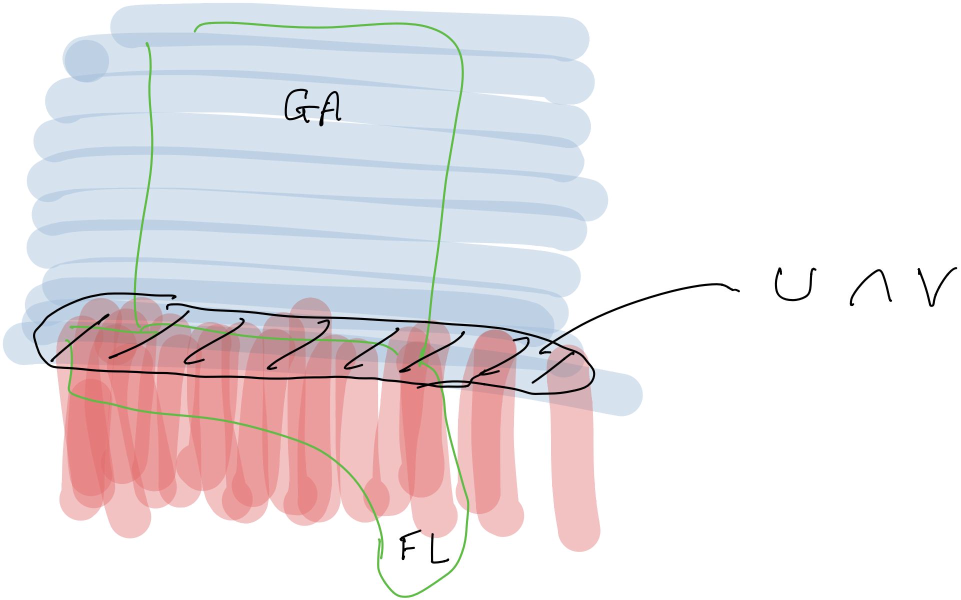
Suppose we have two charts (U, ϕ) and (V, ψ) s.t. $U \cap V \neq \emptyset$.



Two charts are C^∞-related if both $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^∞ on $U \cap V$.

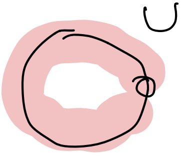
A collection of C^∞-related charts (U_i, ϕ_i) s.t.

$\bigcup_i U_i = M$ is an atlas on M



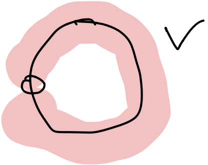
Example SO(2)

$$U = \left\{ \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \in SO(2) \;\middle|\; r_{11} \neq 1 \right\}$$



ϕ_A

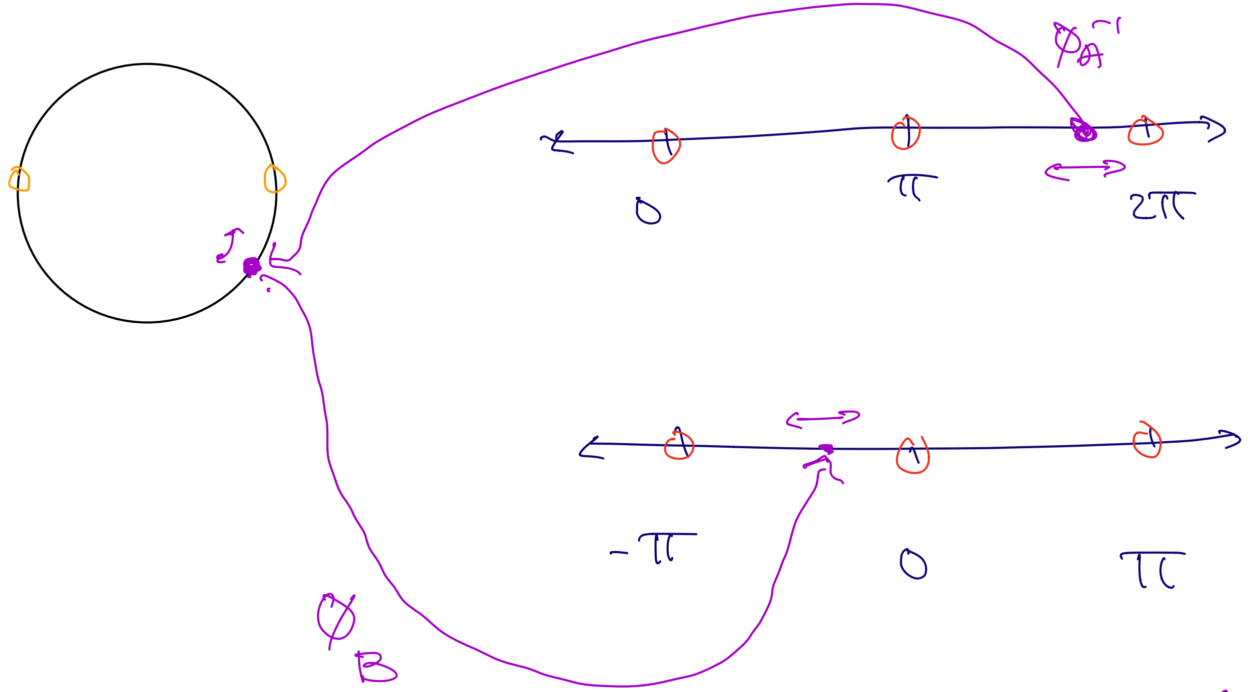
$$V = \left\{ \begin{bmatrix} & \end{bmatrix} \in SO(2) \;\middle|\; r_{11} \neq -1 \right\}$$



ϕ_B

$$\phi_A : U \to (0, 2\pi) \rightsquigarrow X$$

$$\phi_B : V \to (-\pi, \pi) \rightsquigarrow Y$$



$\phi_B \circ \phi_A^{-1}$ is good

Def: A differentiable manifold is a (topological) manifold with an atlas.

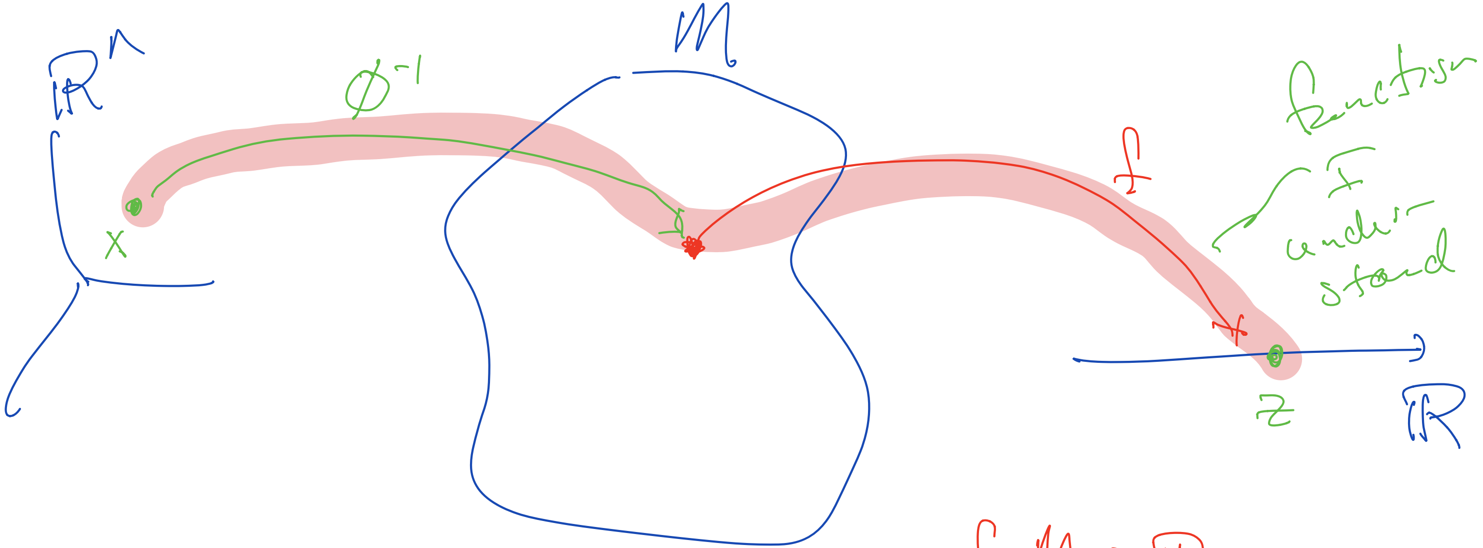
- C^k-related $\Rightarrow$ C^k manifold
- C^∞-related $\Rightarrow$ differentiable or smooth manifold

Calculus on M is difficult.

Calculus on $\mathbb{R}^n$ is easy.

Let $f: M \to \mathbb{R}$, (U, ϕ) a chart....

f is differentiable if $f \circ \phi^{-1}$ is differentiable.



$f: M \to \mathbb{R}$

$\phi^{-1}: \mathbb{R}^n \longrightarrow M$

$$\boxed{f \circ \phi^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}}$$

$$\underbrace{f \circ \phi^{-1}(x)}_{\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2} \dots} = z$$

Def A **Lie group** is a set M and an operation $*$ s.t. M is a differentiable manifold and the group operation is $*$.

Group:

- $x_1, x_2 \in M \Longrightarrow x_1 * x_2 \in M$
- $x_1, x_2, x_3 \in M \Longrightarrow (x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
- $\exists \bar{x}$ s.t. $x * \bar{x} = \bar{x} * x = x$

$\bar{x}$ = identity element

- For all $x \in M$, $\exists x^{-1}$ s.t. $x * x^{-1} = \bar{x}$

$x^{-1} * x = \bar{x}$

$SO(n)$ is a Lie group, matrix mult $\triangleq *$.

because $SO(n)$ is a diff. manifold

+ group properties hold.

For example, $R^T = R^{-1}$

$R_1 R_2 \in SO(2)$

etc.